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Critical Fluctuations for Quantum Mean-Field Models

M. Fannes,[1] A. Kossakowski,[1,3] and A. Verbeure[1]

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We propose a Ginzburg–Landau-type approximation for the local Gibbs states for quantum mean-field models that leads to the exact thermodynamics. Using this approach, we compute the spin fluctuations for some spin-1/2 models. At the critical temperature we find explicitly the distribution function showing abnormal fluctuations.

KEY WORDS: Quantum mean-field models; Ginzburg–Landau theory; critical fluctuations.

1. INTRODUCTION

Systems of dependent random variables ("spins") $x_1,..., x_N$ with joint distributions of the form

$$\exp\left\{\beta N F\left(\frac{x_1 + \cdots x_N}{N}\right)\right\} \prod_{i=1}^{N} \rho(dx_i) \tag{1.1}$$

have extensively been considered in the literature both because of their tractability and their applications to mean-field systems in statistical mechanics. The prototype example is the Curie–Weiss model, where $F(x) = \frac{1}{2}x^2$ and $\rho(\{1\}) = \rho(\{-1\}) = 1/2$.

If such models exhibit a phase transition at a critical value β_c of β, then, for $\beta < \beta_c$, the asymptotic distribution $(N \to \infty)$ of $(1/\sqrt{N})\, S_N$, where $S_N = x_1 + \cdots x_N$, is Gaussian. Ellis and Newman[7,8] showed that this is not true at the critical point $\beta = \beta_c$. In order to compute these critical fluc-

[1] Inst. Theor. Fysica, Universiteit Leuven, Leuven, Belgium.
[2] Bevoegdverklaard Navorser, NFWO Belgium.
[3] On leave from the University of Torun, Poland.

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tuations, one has to take simultaneously the limit $N \to \infty$ for the size of the system and the number of random variables in the fluctuation and S_N has to be scaled down by a factor $N^{3/4}$. This way of taking the limit is related to the idea of finite-size scaling.[6] The result on critical behavior has been generalized and studied from different points of view.[2–4, 11]

In this paper we are interested in the quantum mechanical mean-field models, i.e., we are interested to see what the implications of noncommutativity are on the fluctuations. The random variables become now noncommuting operators. For the sake of transparency we will limit ourselves to the treatment of a few spin-1/2 models. The quantum mechanical result turns out to yield an intrinsically multidimensional version of the classical situation. We examine a number of mean-field models starting from the simplest models of the Curie–Weiss type, over the mean-field Heisenberg model, up to the strongly coupled BCS model. All these models have a phase transition at a nonzero critical temperature T_c and exhibit abnormal fluctuations at T_c. We explicitly compute the corresponding non-Gaussian distribution functions.

Our starting point is based on the well-known de Finetti theorem and on its noncommutative extension[13] that characterize permutation-invariant joint distributions (1.1) of an infinite number of random variables as mixtures of independent, identical distributions. Let $\mathcal{M}_d$ be the algebra of complex $d \times d$ matrices and let $\mathcal{A}$ be the tensor product $\bigotimes^{\mathbb{N}} \mathcal{M}_d$. For $x, y, \ldots \in \mathcal{M}_d$ we denote by $x_i, y_i, \ldots$ copies of $x, y, \ldots$ at the "site" i in $\mathbb{N}$ and we put $S_N(x) = x_1 + \cdots x_N$, $S_N(y) = \cdots$. A mean-field system is then given by specifying Hamiltonians H_N of the type

$$H_N = Nf\left(\frac{1}{N} S_N(x), \frac{1}{N} S_N(y), \ldots\right) \tag{1.2}$$

The popular models are all of this type with f at most quadratic. The equilibrium states of such systems are all expressible as convex combinations of product states $\omega_\rho = \bigotimes_i \rho_i$ on $\mathcal{A}$, where ρ_i is a copy of a density matrix ρ in $\mathcal{M}_d$ satisfying the so-called gap equation[9] (see also ref. 12).

With this in mind and in view of computing fluctuations around equilibrium, we consider the following local approximation σ_N of the density matrix in the volume $\{1, \ldots, N\}$ at the inverse temperature $\beta = 1/kT$:

$$\sigma_N = \frac{\int_{\mathcal{T}_1} m(d\rho) \, [\exp -\beta F_N(\rho)] \bigotimes_{i=1}^N \rho_i}{\int_{\mathcal{T}_1} m(d\rho) \exp -\beta F_N(\rho)} \tag{1.3}$$

where $F_N(\rho)$ is the free energy of the system in the state ω_ρ:

$$\beta F_N(\rho) = \beta \omega_\rho(H_N) + N \operatorname{Tr} \rho \log \rho$$

The measure $m(d\rho)$ is the translation-invariant measure on the set $\mathscr{T}_1$ of density matrices in $\mathscr{M}_d$ and shall be made more explicit later. Because of the extensivity of the free energy $F_N(\rho)$, it is clear that in the limit $N \to \infty$ the canonical Gibbs state, minimizing the free energy density, will be favored and so, for local observables $A \in \mathscr{A}$, $\lim_{N \to \infty} \sigma_N(A)$ coincides with the Gibbs state expectation at inverse temperature β. We will now use formula (1.3) for the computation of fluctuations around equilibrium, i.e., we are interested in the existence and in the explicit expressions of the following characteristic functions:

$$\Phi_x^\beta(\lambda) = \lim_{N \to \infty} \frac{\int m(d\rho) \exp -\beta F_N(\rho) \exp(\lambda/N^\alpha)[\rho(S_N) - \sigma_N(S_N)]}{\int m(d\rho) \exp -\beta F_N(\rho)} \tag{1.4}$$

Of course this limit can only exist and yield a nontrivial result for specific values of the parameter α. If $\alpha = 1/2$, one speaks about normal fluctuations; if $\alpha > 1/2$ one has abnormal fluctuations. The parameter α will in general depend on the temperature and on the observable whose fluctuations are being computed. In fact, formula (1.4) gives the distribution of the fluctuations of the random variables $A_i \mapsto \omega_\rho(A_i)$ on the set of density matrices $\mathscr{T}_1$.

Clearly our approach has been inspired by the Ginzburg–Landau theory of critical phenomena.[5,10] We introduce some randomness around the local restrictions of the equilibrium state, which amounts heuristically to introducing random Hamiltonians $H(\rho) = -\log \rho$ with a probability distribution that is governed by the entropy; this idea goes back to Boltzmann and has been considered by Einstein, Ginzburg, Landau, etc. At this point one should also refer to ref. 1, where fluctuations of the Curie–Weiss model in a random external field are studied.

As far as our results are concerned, our approach yields, both for the classical and the quantum mechanical Curie–Weiss models, the same results for the critical fluctuations as those computed with the local Gibbs states. In these cases all computations can be performed explicitly. It is interesting to remark that the order of criticality of the fluctuations is exactly the same for both the classical and the quantum mechanical cases. The quantum nature appears only in the coefficients governing the distribution. For the Heisenberg and BCS models the computation of the Gibbs critical fluctuations is technically much more involved and has, as far as we know, not been done yet. It is therefore at the moment not possible to compare our results with the distributions obtained from Gibbs states. There is a general belief that the quantum character of the fluctuations is suppressed at the critical point. An argument in favor of this is that critical fluctuations behave like classical random variables. Indeed, the commutator of two critical fluctuations vanishes in the thermodynamic limit:

$$\lim_{N\to\infty}\left[\frac{1}{N^\alpha}S_N(x),\frac{1}{N^\alpha}S_N(y)\right]=0 \qquad \text{if} \quad \alpha>\frac{1}{2}$$

We have, however, no proof for general mean-field models of such a classical behavior.

2. THE MODELS

We will now examine a number of rather simple mean-field models; special attention will be paid to the behavior of the fluctuations at the critical temperature.

2.1. Curie–Weiss Models

We first consider these models on a purely classical level. They are described by a configuration space $\mathscr{K}=K^{\mathbb{N}}$, where $K=\{1,-1\}$. The algebra of observables $\mathscr{A}$ is given by the continuous functions $\mathscr{C}(\mathscr{K})$ on $\mathscr{K}$ and the basic observables are the spin functions σ_i at the different sites: $\sigma_i(\omega)=\omega(i)$, $\omega\in\mathscr{K}$. The local Hamiltonians H_N of these systems are, as in (1.2), given by

$$H_N=-JN\sum_{k=1}^{r}\frac{1}{2k(2k-1)}\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i\right)^{2k}$$

It will be convenient, however, to replace H_N by an equivalent expression [up to order $O(1)$]:

$$H_N=-JN\sum_{k=1}^{r}\frac{1}{2k(2k-1)}\binom{N}{2k}^{-1}\sum_{1\leqslant i_1<\cdots i_{2k}\leqslant N}\sigma_{i_1}\cdots\sigma_{i_{2k}} \tag{2.1}$$

The symmetric product measures of the system are given by single-site measures ρ, which we parametrize with a real parameter $x\in[-1,1]$:

$$\rho=\tfrac{1}{2}(1+x\sigma)$$

Therefore, in this case, the set $\mathscr{T}_1$ in formula (1.3) is the interval $[-1,1]$ and the measure $m(d\rho)$ is the Lebesgue measure dx on $[-1,1]$. The free energy $F_N(\rho)$ equals

$$F_N(\rho)=-N\left(J\sum_{k=1}^{r}\frac{1}{2k(2k-1)}x^{2k}+\frac{1}{\beta}s(\rho)\right)$$

where $s(\rho)$ is the entropy density of the product measure built on ρ:

$$-s(\rho)=\frac{1}{2}(1+x)\log\frac{1}{2}(1+x)+\frac{1}{2}(1-x)\log\frac{1}{2}(1-x)$$

$$=\log 2-\sum_{k=1}^{\infty}\frac{1}{2k(2k-1)}x^{2k}$$

Then

$$\begin{aligned}\beta F_N(\rho)&=\beta N f(\rho)\\&=-N\log 2-N(\beta J-1)\sum_{k=1}^{r}\frac{1}{2k(2k-1)}x^{2k}\\&\quad+N\sum_{k=r+1}^{\infty}\frac{1}{2k(2k-1)}x^{2k}\end{aligned}\tag{2.2}$$

where $f(\rho)$ is the free energy density.

Now we have to compute the limit (1.4) with $S_N=\sigma_1+\cdots\sigma_N$. Clearly the main contribution to the integral will come from the measures ρ that minimize the free energy density $f(\rho)$. It turns out that there is a critical value β_c for β given by $\beta_c J=1$. If $\beta\leqslant\beta_c$, then $f(\rho)$ reaches its minimum for $x=0$. If, on the other hand, $\beta>\beta_c$, then $f(\rho)$ reaches its minimum for $x=\pm x_0$, where x_0 is the positive solution of the equation

$$(\beta J-1)\sum_{k=1}^{r}\frac{1}{2k-1}x^{2k-2}=\sum_{k=r+1}^{\infty}\frac{1}{2k-1}x^{2k-2}$$

We now compute the limit (1.4) for the cases $\beta<\beta_c$ and $\beta=\beta_c$:

(i) If $\beta<\beta_c$, we have normal fluctuations, i.e., $\alpha=1/2$ and

$$\Phi_\sigma^\beta(\lambda)=\frac{\int_{-\infty}^{\infty}dx\exp-\frac{1}{2}(1-\beta J)x^2+\lambda x}{\int_{-\infty}^{\infty}dx\exp-\frac{1}{2}(1-\beta J)x^2}=\exp\frac{1}{2}\frac{\lambda^2}{1-\beta J}$$

(ii) If $\beta=\beta_c$, we have abnormal fluctuations with $\alpha=1-1/[2(r+1)]$ and

$$\Phi_\sigma^{\beta_c}(\lambda)=\frac{\int_{-\infty}^{\infty}dx\exp-[1/(2r+1)(2r+2)]\,x^{2r+2}+\lambda x}{\int_{-\infty}^{\infty}dx\exp-[1/(2r+1)(2r+2)]\,x^{2r+2}}$$

Note that our results coincide completely with the computation of the fluctuations with the local Gibbs states. Remark also that if $\beta=\beta_c$, only the entropy part of the free energy density determines the distribution function of the critical fluctuations. For general mean-field models containing four-point, six-point,..., interactions this will not be true. It is only because of the special choice of the interaction constants in (2.1) that the fluctuations have such a simple distribution functions. Of course, if $\beta < \beta_c$, the distribution of the normal fluctuations is determined by both the internal energy and the entropy.

2.2. Quantum Curie–Weiss Model

This model is quantum mechanical in the sense that the observables are now $\mathscr{A} = \bigotimes^{\mathbb{N}} \mathscr{M}_2$. The local Hamiltonians now read

$$H_N = -\frac{J}{N-1} \sum_{\substack{i,j=1 \\ i<j}}^{N} (\mathbf{e} \cdot \boldsymbol{\sigma}_i)(\mathbf{e} \cdot \boldsymbol{\sigma}_j)$$

where $\mathbf{e}$ is a unit vector in $\mathbb{R}^3$ and $\boldsymbol{\sigma}_i$ is a copy of the spin operators $\boldsymbol{\sigma}$ at the site i. The components $(\sigma_1, \sigma_2, \sigma_3)$ of $\boldsymbol{\sigma}$ are the standard Pauli matrices. This model essentially coincides with the model (2.1) with $r = 1$. In this case the symmetric products states are parametrized by a single-site 2×2 density matrix ρ, which we write as follows:

$$\rho = \frac{1}{2} (\mathbb{1} + \mathbf{x} \cdot \boldsymbol{\sigma}) \tag{2.3}$$

with $\mathbf{x} \in \mathbb{R}^3$, $|\mathbf{x}| \leqslant 1$. The set $\mathscr{T}_1$ of 2×2 density matrices in formula (2.3) is therefore, in this case, isomorphic with the unit sphere in $\mathbb{R}^3$ and we take for the measure $m(d\rho)$ the Lebesgue measure on $\mathbb{R}^3$ restricted to the unit sphere. The free energy density has of course a similar form as that in (2.2):

$$\begin{aligned}\beta F_N(\rho) &= \beta N f(\rho) \\ &= N[-\tfrac{1}{2}\beta J(\mathbf{e} \cdot \mathbf{x})^2 + \tfrac{1}{2}(1 + |\mathbf{x}|) \log \tfrac{1}{2}(1 + |\mathbf{x}|) \\ &\quad + \tfrac{1}{2}(1 - |\mathbf{x}|) \log \tfrac{1}{2}(1 - |\mathbf{x}|)]\end{aligned}$$

An arbitrary local one-site self-adjoint observable can be written in the form $\lambda \mathbf{f} \cdot \boldsymbol{\sigma}$ with $\lambda \in \mathbb{R}$ and $\mathbf{f}$ a unit vector in $\mathbb{R}^3$. The quantity S_N becomes now $S_N = \mathbf{f} \cdot \boldsymbol{\sigma}_1 + \cdots \mathbf{f} \cdot \boldsymbol{\sigma}_N$. As $\rho(S_N) = N \mathbf{f} \cdot x$ and as the measure (1.3) is reflection-invariant, $\sigma_N(S_N) = 0$.

We now compute

$$\Phi^{\beta}_{\mathbf{f} \cdot \boldsymbol{\sigma}}(\lambda) = \lim_{N \to \infty} \frac{\int_{|\mathbf{x}| \leqslant 1} d\mathbf{x} \exp -\beta N f(\rho) + \lambda N^{1-\alpha} \mathbf{f} \cdot \mathbf{x}}{\int_{|\mathbf{x}| \leqslant 1} d\mathbf{x} \exp -\beta N f(\rho)}$$

Again there is a critical point at $\beta_c J = 1$ and the computation yields:

(i) If $\beta < \beta_c$, the limit $N \to \infty$ exists and is nontrivial if $\alpha = 1/2$, i.e., we have normal fluctuations and

$$\Phi^{\beta}_{\mathbf{f}\cdot\boldsymbol{\sigma}}(\lambda) = \frac{\int_{\mathbb{R}^3} d\mathbf{x} \exp -\frac{1}{2}[|\mathbf{x}|^2 - \beta J(\mathbf{e}\cdot\mathbf{x})^2] + \lambda \mathbf{f}\cdot\mathbf{x}}{\int_{\mathbb{R}^3} d\mathbf{x} \exp -\frac{1}{2}[|\mathbf{x}|^2 - \beta J(\mathbf{e}\cdot\mathbf{x})^2]}$$

$$= \exp \frac{1}{2} \lambda^2 \left[\frac{(\mathbf{f}\cdot\mathbf{e})^2}{1-\beta J} + 1 - (\mathbf{f}\cdot\mathbf{e})^2 \right]$$

(ii) If $\beta = \beta_c$, then decompose $\mathbf{f}$ as

$$\mathbf{f} = \mathbf{f}_{\|} + \mathbf{f}_{\perp}; \qquad \mathbf{f}_{\|} = (\mathbf{f}\cdot\mathbf{e})\,\mathbf{e}; \qquad \mathbf{f}_{\perp} = \mathbf{f} - (\mathbf{f}\cdot\mathbf{e})\,\mathbf{e} \tag{2.4}$$

(a) If $\mathbf{f}_{\|} = 0$, then $\alpha = 1/2$ and

$$\Phi^{\beta_c}_{\mathbf{f}\cdot\boldsymbol{\sigma}}(\lambda) = \frac{\int_{-\infty}^{\infty} dx \exp -\frac{1}{2}x^2 + \lambda x}{\int_{-\infty}^{\infty} dx \exp -\frac{1}{2}x^2} = \exp \tfrac{1}{2}\lambda^2$$

(b) If $\mathbf{f}_{\|} \neq 0$, then $\alpha = 3/4$ and

$$\Phi^{\beta_c}_{\mathbf{f}\cdot\boldsymbol{\sigma}}(\lambda) = \frac{\int_{-\infty}^{\infty} dx \exp -(1/12)\,x^4 + \lambda(\mathbf{f}\cdot\mathbf{e})\,x}{\int_{-\infty}^{\infty} dx \exp -(1/12)\,x^4}$$

The computation for the generalized quantum Curie–Weiss models with $r > 1$ [see (2.1)] can be performed in a similar way. Again the results coincide with the fluctuations computed with the local Gibbs or equilibrium states. In this case one has to compute

$$\lim_{N\to\infty} \mathscr{Z}_N^{-1} \operatorname{Tr} \exp \frac{\beta J}{2N} \left(\sum_{i=1}^{N} \mathbf{e}\cdot\boldsymbol{\sigma}_i \right)^2 \exp \frac{\lambda}{N^\alpha} \sum_{i=1}^{N} \mathbf{f}\cdot\boldsymbol{\sigma}_i \tag{2.5}$$

with

$$\mathscr{Z}_N = \operatorname{Tr} \exp \frac{\beta J}{2N} \left(\sum_{i=1}^{N} \mathbf{e}\cdot\boldsymbol{\sigma}_i \right)^2$$

Using the integral representation

$$\exp \tfrac{1}{2}y^2 = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dx \exp -\tfrac{1}{2}x^2 \pm yx$$

formula (2.5) becomes

$$\lim_{N\to\infty} \mathscr{Z}_N^{-1} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} dx \exp -\tfrac{1}{2}x^2 \left[\operatorname{Tr} \exp \left(\frac{\beta J}{N} \right)^{1/2} x\mathbf{e}\cdot\boldsymbol{\sigma} \exp \frac{\lambda}{N^\alpha} \mathbf{f}\cdot\boldsymbol{\sigma} \right]^N \tag{2.6}$$

and

$$\operatorname{Tr} \exp \left(\frac{\beta J}{N}\right)^{1/2} x \mathbf{e} \cdot \boldsymbol{\sigma} \exp \frac{\lambda}{N^\alpha} \mathbf{f} \cdot \boldsymbol{\sigma}$$

$$= 2\left[\cosh \left(\frac{\beta J}{N}\right)^{1/2} x \cosh \frac{\lambda}{N^\alpha} + \mathbf{f} \cdot \mathbf{e} \sinh \left(\frac{\beta J}{N}\right)^{1/2} x \sinh \frac{\lambda}{N^\alpha}\right]$$

This reduces formula (2.6) to an easily computable limit, yielding the same results as above.

2.3. Mean-Field Heisenberg Model

The Hamiltonian of the system is now

$$H_N = -\frac{J}{N-1} \sum_{\substack{i,j=1 \\ i<j}}^{N} \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j$$

We proceed as in Section 2.2 for the labeling of the density matrices ρ [see (2.3)]. The free energy becomes

$$\begin{aligned} \beta F_N(\rho) &= \beta N f(\rho) \\ &= N[-\tfrac{1}{2}\beta J\, |\mathbf{x}|^2 + \tfrac{1}{2}(1+|\mathbf{x}|) \log \tfrac{1}{2}(1+|\mathbf{x}|) \\ &\quad + \tfrac{1}{2}(1-|\mathbf{x}|) \log \tfrac{1}{2}(1-|\mathbf{x}|)] \end{aligned}$$

We compute again the fluctuation of the magnetization in the direction of a unit vector $\mathbf{f}$ in $\mathbb{R}^3$. This leads to the computation of

$$\lim_{N\to\infty} \frac{\int_{|\mathbf{x}|\leqslant 1} d\mathbf{x} \exp -\beta N f(\rho) + \lambda N^{1-\alpha} \mathbf{f} \cdot \mathbf{x}}{\int_{|\mathbf{x}|\leqslant 1} d\mathbf{x} \exp -\beta N f(\rho)} \tag{2.7}$$

The critical temperature is given by the equation $\beta_c J = 1$. If $\beta \leqslant \beta_c$, then $\mathbf{x} = 0$ is the only solution minimizing the free energy density $f(\rho)$ and the expansion of f around $\mathbf{x} = 0$ is given by

$$-\beta f(\rho) = \log 2 - \tfrac{1}{2}(1-\beta J)\, |\mathbf{x}|^2 - \tfrac{1}{12}\, |\mathbf{x}|^4 + O(|\mathbf{x}|^6)$$

If $\beta > \beta_c$, then the minimum of $f(\rho)$ is reached at all nonzero solutions of the equation $|\mathbf{x}| = \tanh \beta J\, |\mathbf{x}|$. The fluctuations for $\beta < \beta_c$ are again normal. We limit ourselves to the case $\beta = \beta_c$. The limit in formula (2.7) can be computed with $\alpha = 3/4$ and yields in a straightforward way the following distribution:

$$\Phi^{\beta_c}_{\mathbf{f}\cdot\sigma}(\lambda) = \frac{\int_{\mathbb{R}^3} d\mathbf{x} \exp -(1/12)\,|\mathbf{x}|^4 + \lambda \mathbf{f}\cdot\mathbf{x}}{\int_{\mathbb{R}^3} d\mathbf{x} \exp -(1/12)\,|\mathbf{x}|^4}$$
$$= \frac{\int_0^\infty r^2\, dr [\exp -(1/12)\, r^4] \sinh(\lambda r/\lambda r)}{\int_0^\infty r^2\, dr \exp -(1/12)\, r^4}$$

2.4. The BCS Model

Finally we compute the fluctuations for the BCS model:

$$H_N = \sum_{i=1}^{N} \sigma_{3,i} - \frac{2\gamma}{N-1} \sum_{\substack{i,j=1\\ i<j}}^{N} (\sigma_i^+ \sigma_j^- + \sigma_i^- \sigma_j^+)$$

where $\gamma > 1$ and $\sigma^\pm = \frac{1}{2}(\sigma_1 \pm \sigma_2)$. The critical temperature β_c is now given by $\gamma \tanh \beta_c = 1$. For $\beta < \beta_c$ the fluctuations are again normal. We will limit ourselves to the case $\beta = \beta_c$. The minimum of the free energy density $f(\rho)$ is reached for a 2×2 density matrix parametrized by $\mathbf{x}_0 = (0, 0, 1/\lambda)$. The free energy density is again analytic around that point and, introducing the parameters

$$x_1 = r\cos\varphi, \qquad x_2 = r\sin\varphi, \qquad x_3 = z + \frac{1}{\gamma}$$

we find the leading terms in the expansion of the free energy density

$$\beta f(\rho) = \beta f(\rho(\mathbf{x}_0)) + Az^2 + Br^2 z + Cr^4 + \cdots$$

where the coefficients A, B, C are given by the following expressions: denote $U(p) = -(1/p)\,\dot{s}(p)$, where s is the entropy function; then

$$A = \frac{1}{2}\left[U\left(\frac{1}{\gamma}\right) + \frac{1}{\gamma}\,\dot{U}\left(\frac{1}{\gamma}\right)\right]$$
$$B = \frac{1}{2}\,\dot{U}\left(\frac{1}{\gamma}\right)$$
$$C = \frac{1}{8}\,\gamma \dot{U}\left(\frac{1}{\gamma}\right)$$

The quadratic form $Az^2 + Br^2 z + Cr^4$ in z and r^2 is positive because

$$B^2 - 4AC = \frac{1}{4}\,\gamma U \dot{U}\left(\frac{1}{\gamma}\right) < 0$$

As in formula (2.4), we have to consider two cases:

(a) If $\mathbf{f}\cdot\boldsymbol{\sigma} = \pm\sigma_3$, then the distribution becomes

$$\Phi^{\beta_c}_{\sigma_3}(\lambda) = \lim_{N\to\infty} \frac{\int_{|\mathbf{x}|\leqslant 1} d\mathbf{x} \exp -N(Az^2 + Br^2z + Cr^4) + N^{1-\alpha}\lambda z}{\int_{|\mathbf{x}|\leqslant 1} d\mathbf{x} \exp -N(Az^2 + Br^2z + Cr^4)}$$

This limit will exist and be nontrivial if $\alpha = 1/2$; introducing the scalings $N^{1/2}z$ and $N^{1/4}r$, one gets

$$\Phi^{\beta_c}_{\sigma_3}(\lambda) = \frac{\int_{-\infty}^{\infty} dz \int_0^\infty r\, dr \exp -(Az^2 + Br^2z + Cr^4) + \lambda z}{\int_{-\infty}^{\infty} dz \int_0^\infty r\, dr \exp -(Az^2 + Br^2z + Cr^4)}$$

(b) If $(f_3)^2 < 1$, the distribution $\Phi^{\beta_c}_{\mathbf{f}\cdot\boldsymbol{\sigma}}(\lambda)$ exists and is nontrivial if $\alpha = 3/4$; using the same scalings as above, one gets

$$\Phi^{\beta_c}_{\mathbf{f}\cdot\boldsymbol{\sigma}}(\lambda)$$

$$= \frac{\int_{-\infty}^{\infty} dz \int_0^\infty r\, dr \int_0^{2\pi} d\theta\, [\exp -(Az^2 + Br^2z + Cr^4) + \lambda(1-f_3^2)^{1/2}\, r\cos\theta]}{2\pi \int_{-\infty}^{\infty} dz \int_0^\infty r\, dr \int_0^{2\pi} d\theta \exp -(Az^2 + Br^2z + Cr^4)}$$

$$= \frac{\int_0^\infty r\, dr \int_0^{2\pi} d\theta \exp -\frac{1}{4}[(4AC - B^2)/A]\, r^4 + \lambda(1-f_3^2)^{1/2}\, r\cos\theta}{2\pi \int_0^\infty r\, dr \int_0^{2\pi} d\theta \exp -\frac{1}{4}[(4AC - B^2)/A]\, r^4}$$

Although we considered here only a couple of simple mean-field models, already quite different distribution functions have been obtained and the multidimensional character clearly emerges. It is clear that by considering more complicated models, like the one in (2.1), it will also be possible to increase the order α of the critical fluctuations for quantum mechanical models. It is also clear that by considering higher-point interactions or by increasing the single-site spin the complexity of the critical distribution functions increases.

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